Noncommutative $U(1)$ Gauge Theory As a Non-Linear Sigma Model

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Abstract

Noncommutative $U(1)$ gauge theory in $4-$dimensions is shown to be equivalent in some scaling limit to an ordinary non-linear sigma model in $2-$dimensions . The model in this regime is solvable and the corresponding exact beta function is found. We also show that classical $U(n)$ gauge theory on $\mathbf{R}^{d-2} \times \mathbf{R}_\theta^2$ can be approximated by a sequence of ordinary $(d-2)-$dimensional Georgi-Glashow models with gauge groups $U(n(L+1))$ where $L+1$ is the matrix size of the regularized noncommutative plane $\mathbf{R}_\theta^2$.

1 Introduction

The Moyal-Weyl noncommutative space is a $0-$dimensional matrix model and thus it is not a continuum manifold . It is known that this space can be represented by infinite dimensional matrices acting on some infinite dimensional Hilbert space [1]. The fuzzy sphere on the other hand although it is a $0-$dimensional matrix model it acts on a finite dimensional Hilbert space [2, 9, 11, 13]. In other words the fuzzy sphere can be represented by finite dimensional matrices .Clearly and in analogy with the continuum situation one should be able to get form one space to the other and vice versa. However and as it turns out we have more structure in this case since the fuzzy sphere can in fact be thought of as a regularization of the noncommutative plane [12].

The noncommutative plane is given in terms of the algebra of the harmonic oscillator. The coordinates on this space are given by $x_1 = \sqrt{\frac{\theta}{2}}(a + a^+)$, $x_2 = -i\sqrt{\frac{\theta}{2}}(a - a^+)$, $[x_1, x_2] = i\theta$. The algebra on the fuzzy sphere is given on the other hand in terms of the generators L_a of the IRR $\frac{L}{2}$ of the $SU(2)$ Lie algebra. The global coordinates on the fuzzy sphere are defined by $x_a = \frac{RL_a}{|L|}$, $[x_a, x_b] = \frac{iR}{|L|}\epsilon_{abc}x_c$, $\sum_a x_a^2 = R^2$.

It is not difficult to argue from the above equations that the fuzzy sphere algebra is nothing else but a deformation of the Moyal-Weyl plane algebra which results in a finite dimensional Hilbert space [10]. Taking L to infinity reduces the fuzzy sphere to a noncommutative plane. This cut-off is gauge invariant as one can also see from the action. $U(1)$ gauge action on the noncommutative Moyal-Weyl plane is given by [1]

$$S_\theta = \frac{\theta^2}{4g^2}Tr\hat{F}_{ij}^2 = \frac{\theta^2}{4g^2}Tr\sum_{i,j}\left(i[\hat{D}_i, \hat{D}_j] - \frac{1}{\theta^2}(\epsilon^{-1})_{ij}\right)^2, \tag{1.1}$$

where $\hat{D}_i$ is the covariant derivative define by $\hat{D}_i = -\frac{1}{\theta^2}(\epsilon^{-1})_{ij}\hat{x}_j + \hat{A}_i$ and $\hat{F}_{ij}$ is the curvature tensor. Tr is over the infinite dimensional Hilbert space H . This theory can be regularized by the following finite dimensional matrix model [10]

$$S_{L,R} = \frac{R^2}{4g^2}\frac{1}{L+1}Tr_L F_{ab}^2 = \frac{R^2}{4g^2}\frac{1}{L+1}Tr_L \sum_{a,b}\left(i[D_a, D_b] + \sum_c \frac{1}{R}\epsilon_{abc}D_c\right)^2, \qquad (1.2)$$

with the constraint $[3, 4, 7]$

$$D_a D_a = \frac{|L|^2}{R^2} \;,\; |L|^2 = \frac{L}{2}\left(\frac{L}{2} + 1\right). \qquad (1.3)$$

The equations of motion are given by $2R[F_{cb}, D_b] = i\epsilon_{abc}F_{ab}$. They are solved by the zero-curvature condition $F_{ab} = 0$ which are equivalent to $D_a = \frac{1}{R}L_a$. This is exactly the fuzzy sphere [6]. Expanding around this solution by writing $D_a = \frac{1}{R}L_a + A_a$ leads to $U(1)$ gauge theory on the fuzzy sphere . The above constraint is needed to describe a $2-$dimensional gauge field and also to stabilize the fuzzy sphere solution [4]. This is also related to the fact that $a = 1, 2, 3$ since the differential calculus on the fuzzy sphere is $3-$dimensional. Tr_L is a finite dimensional trace over the Hilbert space H_L , for example $Tr 1 = L + 1$.

We are interested therefore in a continuum double scaling limit of large R and large L taken together (restricting the theory around the north pole for example) as follows $[6, 12]$

$$R, L \longrightarrow \infty \;;\; \text{keeping } \frac{R^2}{|L|^{2q}} = \text{ fixed} \equiv \theta^2 \;,\; q = \text{real number} \qquad (1.4)$$

The action (1.2) is seen to tend to (1.1) with a resulting effective noncommutativity

$$\theta_{eff}^2 = \theta^2\xi^2 \;,\; \xi^2 = |L|^{2q-2}(L+1) \qquad (1.5)$$

For $q > \frac{1}{2}$, $\xi^2 \longrightarrow \infty$ when $L \longrightarrow \infty$ and thus θ_{eff} corresponds to strong noncommutativity. For $q < \frac{1}{2}$ we find that $\xi^2 \longrightarrow 0$ when $L \longrightarrow \infty$ and θ_{eff} corresponds to weak noncommutativity. For $q = \frac{1}{2}$ the effective noncommutativity parameter is exactly given by $\theta_{eff}^2 = 2\theta^2$. This statement can be made precise using the coherent states approach [5].

On all noncommutative spaces it is always possible to map operators $\hat{O}$ to fields $O(x)$ using the so-called Weyl map. The pointwise multiplication of operators will be replaced by a star product while traces will be replaced by ordinary integrals . For example on the noncommutative Moyal-Weyl plane the $U(1)$ action (1.1) can be rewritten in this language as follows

$$S_\theta = \frac{1}{4g^2}\int d^2x F_{ij}^2 \;,\; F_{ij} = \partial_i A_j - \partial_j A_i + i\{A_i, A_j\}_*. \qquad (1.6)$$

2 The Noncommutative $U(1)$ Theory In $4-$Dimensions

The action in higher dimensions is similar to (1.6). In $\mathbf{R}_\theta^d = \mathbf{R}^{d-2}\times\mathbf{R}_\theta^2$ we have the commutation relations

$$[x_\mu, x_\nu] = 0 \;,\; [x_\mu, x_i] = 0 \;,\; [\hat{x}_i, \hat{x}_j] = i\theta^2\epsilon_{ij} \;,\mu, \nu = 1, ..., d-2 \;,\; i, j = d-1, d. \qquad (2.1)$$

2

For simplicity we are only considering minimal noncommutativity where only two spatial coordinates fail to commute. In order to maintain unitarity of the quantum theory we are also assuming that the time direction lies in the commutative submanifold . The covariant derivatives in this case are given by $\hat{D}_\mu = -i\partial_\mu + \hat{A}_\mu$, $\hat{D}_i = -\frac{1}{\theta^2}(\epsilon^{-1})_{ij}\hat{x}_j + \hat{A}_i$ and the $U(1)$ action reads exactly like (1.6) where the star product is now given in terms of the commutation relations (2.1).

This action can be reexpressed back in terms of operators as follows

$$S_\theta \equiv \frac{\theta^2}{4g^2}\int d^{d-2}x Tr\hat{F}_{\mu\nu}^2 + \frac{\theta^2}{2g^2}\int d^{d-2}x \sum_{i=d-1}^{d} Tr\hat{F}_{\mu i}^2 + \frac{\theta^2}{4g^2}\int d^{d-2}x \sum_{i,j=d-1}^{d} Tr\hat{F}_{ij}^2. \quad (2.2)$$

In above we have deliberately used the fact that we can replace the integral over the noncommutative directions x_{d-1} and x_d by a trace over an infinite dimensional Hilbert space by using the Weyl Map. By doing this we have therefore also replaced the underlying star product of functions by pointwise multiplication of operators. The trace Tr is thus associated with the two noncommutative coordinates x_{d-1} and x_d . The model looks very much like a $U(\infty)$ gauge theory on $\mathbf{R}^{d-2}$ with a Higgs particle in the adjoint of the group.

In the remainder of this section we will confine ourselves to $4-$dimensions. From equation (2.2) we can see that for each point of the $2-$dimensional commutative $\mathbf{R}^2$ the above action is an infinite dimensional matrix model. It can be regularized if we approximate the noncommutative plane by a fuzzy sphere in exactly the same way as before. The regularized action reads [10]

$$S_{\theta;L} = \frac{1}{4\lambda^2}\int d^2x Tr_L \mathcal{F}_{\mu\nu}^2 - \frac{1}{2\lambda^2}\int d^2x \sum_{a=1}^{3} Tr_L[\mathcal{D}_\mu, D_a]^2 - \frac{1}{4\lambda^2}\int d^2x V(D_a). \quad (2.3)$$

D_a are $(L+1)\times(L+1)$ matrices which are fields on $\mathbf{R}^2$ and satisfy

$$D_a^2 = \frac{|L|^2}{R^2}. \quad (2.4)$$

The potential term is

$$V(D_a) = Tr_L[D_a, D_b]^2 - \frac{2i}{R}\epsilon_{abc}Tr_L[D_a, D_b]D_c - \frac{2}{R^4}(L+1)|L|^2, \quad (2.5)$$

while the coupling constant is $\lambda^2 = \frac{g^2(L+1)}{R^2}$ where g is the coupling constant on the noncommutative space $\mathbf{R}_\theta^4$.

$$\mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu] \ , \ \mathcal{D}_\mu = -i\partial_\mu + \mathcal{A}_\mu. \quad (2.6)$$

$\mathcal{A}_\mu$ are $(L+1)\times(L+1)$ matrices which are fields on $\mathbf{R}^2$. This is clearly a $U(L+1)$ gauge theory with adjoint matter , i.e the original noncommutative degrees of freedom are traded for ordinary color degrees of freedom. The field $\mathcal{A}_\mu$ can be separated into a $U(1)$ gauge field a_μ and an $SU(L+1)$ gauge field A_μ as follows

$$\mathcal{A}_\mu(x) = a_\mu(x)\mathbf{1} + A_\mu(x) \ , \ A_\mu(x) = A_{\mu A}(x)T_A. \quad (2.7)$$

3

Similarly we write

$$D_a = n_a + \Phi_a \ , \ \ \Phi_a = \Phi_{aA}T_A, \tag{2.8}$$

Under gauge transformations n_a is a singlet while Φ_a transforms in the adjoint representation of the non-abelian group $SU(L+1)$. These are "scalars" with respect to the commutative directions of $\mathbf{R}_\theta^4 = \mathbf{R}_\theta^2 \times \mathbf{R}^2$. The abelian $U(1)$ field a_μ is found from the action to be free and thus it can be integrated out . The non-abelian $SU(L+1)$ field is seen to be defined on a two dimensional spacetime and thus it can also be integrated out if one uses the light-cone gauge. To this end we rotate first to Minkowski signature then we fix the $SU(L+1)$ symmetry by going to the light-cone gauge given by

$$A_- = 0 \Leftrightarrow A_1 = A_2 = \sqrt{2}\lambda A_+. \tag{2.9}$$

The integral over the A_+ field becomes Gaussian and thus it can be easily done . It gives a non-local Coulomb interaction between the Φ_{aC} fields. We define

$$\Delta_{AB}(x,y) = -\frac{\delta_{AB}}{2}|x_- - y_-|\delta(x_+ - y_+) \ , \ \ f_{ABC}(\partial_-\Phi_{aA})\Phi_{aC} \equiv (\vec{\Phi}_a \times_L \partial_- \vec{\Phi}_a)_B$$

where Δ_{AB} is clearly the propagator of the Φ_{aC} fields and then write the final result in the form

$$
\begin{aligned}
\hat{S}_{\theta;L} &= \frac{N}{2\lambda^2}\int d^2x(\partial_\mu n_a)(\partial^\mu n_a) + \frac{1}{4\lambda^2}\int d^2x(\partial_\mu\Phi_{aA})(\partial^\mu\Phi_{aA}) \\
&- \frac{1}{4\lambda^2}\int d^2x d^2y(\vec{\Phi}_a \times_L \partial_-\vec{\Phi}_a)_A(x)\Delta_{AB}(x,y)(\vec{\Phi}_b \times_L \partial_-\vec{\Phi}_b)_B(y) - \frac{1}{4\lambda^2}\int d^2xV(\Phi_a).
\end{aligned}
\tag{2.10}
$$

The constraint $D_aD_a = \frac{|L|^2}{R^2}$ can be rewritten in the form

$$n_a^2 + \frac{1}{2(L+1)}\Phi_{aA}^2 = \frac{|L|^2}{R^2} \ , \ \ n_a\Phi_{aC} + \frac{1}{4}d_{ABC}\Phi_{aA}\Phi_{aB} = 0, \tag{2.11}$$

From the structure of this constraint and from the action we can see that the field n_a appears at most quadratically. The corresponding path integral can be done exactly in the large L limit and one obtains (with $\chi_{aA} = \frac{R}{|L|}\frac{1}{\sqrt{2N}}\Phi_{aA}$) the reduced action [10]

$$\bar{S}_{\theta;L} = \frac{1}{4\bar{\lambda}^2}\int d^2x(\partial_\mu\chi_{aA})(\partial^\mu\chi_{aA}) - \frac{|L|^2(L+1)}{2\bar{\lambda}^2 R^2}\int d^2x\bar{V}(\chi_a) \ , \ \ \bar{\lambda}^2 = \frac{g^2}{2|L|^2} \tag{2.12}$$

where

$$
\begin{aligned}
\bar{V}(\chi_a) &= \int d^2y(\vec{\chi}_a \times_L \partial_-\vec{\chi}_a)_A(x)\Delta_{AB}(x,y)(\vec{\chi}_b \times_L \partial_-\vec{\chi}_b)_B(y) + Tr_L[\chi_a,\chi_b]^2 \\
&- \frac{2i}{|L|\sqrt{2(L+1)}}\epsilon_{abc}Tr_L[\chi_a,\chi_b]\chi_c - \frac{1}{2|L|^2(L+1)}.
\end{aligned}
$$

4

The fields χ_{aA} satisfy now the constraints

$$\chi_{aA}^2 = 1 \ , \ d_{ABC}\chi_{aA}\chi_{aB} = -\frac{2e_a R}{\sqrt{2|L|^2(L+1)}}\chi_{aC}, \tag{2.13}$$

where e_a is an arbitrary constant vector in $\mathbf{R}^3$ [10]. Since $R^2 = \theta^2|L|^{2q}$ the overall coupling in front of the potential $\bar{V}$ behaves as

$$\frac{|L|^2(L+1)}{2\bar{\lambda}^2 R^2} \sim \frac{1}{\bar{\lambda}^2 \theta^2}\Big(\frac{L}{2}\Big)^{3-2q} \tag{2.14}$$

Thus for all scalings with $q > \frac{3}{2}$ this potential can be neglected compared to the kinetic term. The fuzzy theory for these scalings becomes a theory living on a noncommutative plane with effective deformation parameter

$$\theta_{eff}^2 \sim 2\theta^2\Big(\frac{L}{2}\Big)^{2q-1} \tag{2.15}$$

We are therefore probing the strong noncommutativity region of the Moyal-Weyl model . The partition function in this case is given by

$$Z = \int \mathcal{D}J\mathcal{D}J_C e^{i\int d^2xJ}exp\Big(-\frac{3}{2}TRlogD\Big)exp\Big(-\bar{e}^2\theta^2|L|^{2(q-\frac{3}{2})}\int d^2xd^2yJ_A(x)D_{AB}^{-1}(x,y)J_B(y)\Big)$$

where $D(=D_{AB}(x,y))$ is the Laplacian

$$D_{AB}(x,y) = \delta^2(x-y)\Big(-\frac{1}{4\bar{\lambda}^2}\partial^2\delta_{AB} + iJ\delta_{AB} + iJ_Cd_{ABC}\Big). \tag{2.16}$$

At this stage it is obvious that in the large L limit only configurations where $J_A = 0$ are relevant and thus one ends up with the partition function

$$Z = \int \mathcal{D}J exp\Big(\frac{i}{4\bar{\lambda}^2}\int d^2xJ - \frac{M}{2}\int d^2x <x|log\big(-\partial^2 + iJ\big)|x>\Big). \tag{2.17}$$

This is exactly the partition function of an $O(M)$ non-linear sigma model in the limit $M\longrightarrow\infty$ with $\bar{\lambda}^2M$ held fixed equal to $\bar{\lambda}^2M = 6g^2$ where $M = 3(N^2-1) = 3L(L+2)$. All terms in the exponent are now of the same order M and thus the model can be solved using steepest descents . For example we can derive the beta function [8]

$$\beta(g_r) = \mu\frac{\partial g_r}{\partial \mu} = -\frac{3}{\pi}g_r^3. \tag{2.18}$$

This result agrees nicely with the one-loop calculation of the beta function of $U(1)$ theory on the Moyal-Weyl Plane [1]. The crucial difference is the fact that this result is exact to all orders in $\bar{\lambda}^2M = 6g^2$ and thus it is intrinsically nonperturbative [8].

3 $U(n)$ Gauge Theory and The Presnajder-Steinacker Action

$U(n)$ gauge theory on $\mathbf{R}^{d-2} \times \mathbf{R}_\theta^2$ is given by the action

$$S_\theta = \frac{1}{4g^2} \int d^d x \sum_{A,B=1}^{d} tr_n F_{AB}^2 = \frac{\theta^2}{4g^2} \int d^{d-2}x Tr tr_n \hat{F}_{AB}^2 \ , \ \ A, B = 1, ..., d, \tag{3.1}$$

with $F_{AB} = \partial_A A_B - \partial_B A_A + i\{A_A, A_B\}_*$ and $\hat{F}_{AB} = \hat{\partial}_A \hat{A}_B - \hat{\partial}_B \hat{A}_A + i[\hat{A}_A, \hat{A}_B]$. The regularized theory can be obtained in the same way as before and one ends up with the equations (2.3), (2.4) , (2.5) and (2.6) with the only replacement $Tr_L \longrightarrow Tr_N = Tr_L tr_n$. This is clearly a $U(N+1) \equiv U(n(L+1))$ gauge theory which has also the interpretation of being a $U(n)$ gauge theory on $\mathbf{R}^{d-2} \times \mathbf{S}_L^2$. The corresponding action can be simplified further if one uses the following trick due to Presnajder [14] and Steinacker [7].

The 3−matrix action $S_{L,R}$ given in equation (1.2) togther with the constraint (1.3) can be derived from a much simpler 1−matrix model . To this end we introduce Pauli matrices σ_a and we write the operator

$$\bar{\bar{\Phi}} = (\frac{1}{2} + \sigma_a L_a) \otimes \mathbf{1}_n. \tag{3.2}$$

This is a $2N$−dimensional matrix. It is a trivial exrecise to check that $\bar{\bar{\Phi}} = (j(j+1) - (\frac{L+1}{2})^2) \otimes \mathbf{1}_n$ where j is the eigenvalue of the operator $\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}$ which takes the two values $\frac{L+1}{2}$ and $\frac{L-1}{2}$. The eigenvalues of $\bar{\bar{\Phi}}$ are therefore $\frac{L+1}{2}$ with multiplicity $n(L+2)$ and $-\frac{L+1}{2}$ with multiplicity nL. As it turns out this matrix $\bar{\bar{\Phi}}$ can be obtained as a classical configuration of the following $2N$−dimensional 1−matrix action

$$S[\Phi] = \frac{1}{4g^2 R^2} \frac{1}{L+1} Tr_N tr_2 \left[\frac{(L+1)^4}{16} - \frac{(L+1)^2}{2} \Phi^2 + \Phi^4 \right]. \tag{3.3}$$

Indeed the equations of motion derived from this action reads

$$\Phi(\Phi^2 - \frac{(L+1)^2}{4}) = 0.$$

It is easy to see that $\bar{\bar{\Phi}}$ solves this equation of motion and that the value of the action in this configuration is identically zero , i.e $S[\Phi = \bar{\bar{\Phi}}] = 0$. In general and in terms of the eigenvalues ϕ_i of Φ this equation reads $\phi_i(\phi_i^2 - \frac{(L+1)^2}{4}) = 0$ which means that $\phi_i = 0, +\frac{L+1}{2}, -\frac{L+1}{2}$, i.e classical configurations are matrices of eigenvalues 0 , $+\frac{L+1}{2}$ and $-\frac{L+1}{2}$ with corresponding multiplicities n_0 , n_+ and n_- respectively which must clearly add up to $n_0 + n_+ + n_- = 2n(L+1)$. The action for each zero eigenvalue $\phi_i = 0$ is given by $S[\phi_i = 0] = \frac{1}{4g^2 R^2} \frac{1}{L+1} \frac{(L+1)^4}{16}$ which is suppressed in the large L limit and thus these stationary points do not contribute in the large L limit. Expanding around the vacuum $\bar{\phi}$ by writing

$$\Phi = \frac{1}{2} + \rho + R\sigma_a D_a \tag{3.4}$$

6

where ρ and D_a are $N \times N$ matrices will immediately lead to the action $S_{L,R}$ if one also imposes the condition [7]

$$\rho = 0. \tag{3.5}$$

Indeed we find explicitly

$$S[\Phi = \frac{1}{2} + R\sigma_a D_a] = \frac{1}{4g^2}\frac{R^2}{L+1}Tr_L tr_n\left[F_{ab}^2 + 2(D_a^2 - \frac{|L|^2}{R^2})^2\right] \ , \ F_{ab} = i[D_a, D_b] + \frac{1}{R}\epsilon_{abc}D_c \tag{3.6}$$

The second term in this action can be shown to implement exactly the constraint (1.3) as we want. The $U(n)$ action on $\mathbf{R}^{d-2} \times \mathbf{S}_L^2$ can then be taken (without any further constraint) to be

$$S_{\theta;L} = \frac{1}{4\lambda^2}\int d^{d-2}x Tr_N \mathcal{F}_{\mu\nu}^2 - \frac{1}{2\lambda^2}\int d^{d-2}x \sum_{a=1}^{3} Tr_N[\mathcal{D}_\mu, D_a]^2 - \frac{1}{4\lambda^2}\int d^{d-2}x W(D_a) \tag{3.7}$$

where the potential reads now as follows

$$W(D_a) = -Tr_N\left(F_{ab}^2 + 2(D_a^2 - \frac{|L|^2}{R^2})^2\right). \tag{3.8}$$

In terms of the scalar field Φ we can rewrite this action as follows

$$\begin{aligned}
S_{\theta;L} &= \frac{1}{4\lambda^2}\int d^{d-2}x Tr_N \mathcal{F}_{\mu\nu}^2 + \frac{1}{4\lambda^2 R^2}\int d^{d-2}x Tr_N[\mathcal{D}_\mu, \Phi_{ij}]^+[\mathcal{D}_\mu, \Phi_{ij}] \\
&+ \frac{1}{4R^4\lambda^2}\int d^{d-2}x Tr_N\left(\Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li} - \frac{(L+1)^2}{2}\Phi_{ij}\Phi_{ji} + \frac{(L+1)^4}{16}\right). \tag{3.9}
\end{aligned}$$

Recall that Φ is a 2×2 matrix where each component Φ_{ij} is an $N \times N$ matrix . Under $U(N)$ gauge transformations each of these components transforms covariantly . In deriving the above result we have used the Fierz identity $(\sigma_a)_{ij}(\sigma_a)_{kl} = 2(\delta_{il}\delta_{kj} - \frac{1}{2}\delta_{ij}\delta_{kl})$ as well as the constraint (3.5) which we can write in the equivalent form

$$\Phi_{11} + \Phi_{22} = 1. \tag{3.10}$$

The action (3.9) is essentially a Georgi-Glashow model with several scalar fields $\frac{\Phi_{ij}}{R}$ in the adjoint representation of the group which are restricted to satisfy the constraint (3.10). In other words $U(n)$ gauge theory on $\mathbf{R}^{d-2} \times \mathbf{R}^2_{\theta_{eff}}$ can be approximated using a fuzzy sphere of matrix size $L+1$ and radius R by a sequence of Georgi-Glashow models given by (3.9)+(3.10) with increasing L and R . The gauge groups are seen to be given by $U(n(L+1))$ while the coupling constants are given by $\lambda^2 \sim \frac{g^2}{\theta_{eff}^2}$ where g^2 is the coupling constant on $\mathbf{R}^{d-2} \times \mathbf{R}^2_{\theta_{eff}}$. The noncommutativity parameter on $\mathbf{R}^{d-2} \times \mathbf{R}^2_{\theta_{eff}}$ is found to be given by $\theta_{eff}^2 \sim 2\theta^2(\frac{L}{2})^{2q-1}$ where $\theta^2 = R^2/|L|^{2q}$ is always kept fixed. Clearly the quantum theory depends on the way we take the limit. We defer the study of these models to a future correspondence.

4   Conclusion

In this article we have considered gauge theory on $\mathbf{R}^{d-2} \times \mathbf{R}^2_\theta$. We have regularized the two non-commuting directions by replacing them with a fuzzy sphere . This turns the noncommutative field theory into an ordinary commutative field theory amenable to the standard techniques of quantization and renormalization , etc. The non-trivial ingredient in this construction remains always the definition of the limit which requires in our opinion further study . The $U(1)$ theory is seen in some scaling limit to correspond to an ordinary $2d$ non-linear sigma model thus allowing us to derive the beta function of the theory . The result agrees with perturbation theory but the question remains what happens in other scaling limits. Higher $U(n)$ in higher dimensions are found to be classically equivalent to a sequence of Georgi-Glashow models defined on the commutative submanifold. Their quantum properties will however be studied elsewhere.

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